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# Bernstein Type Theorems for Compact Sets in $\mathbb{R}^n$

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In this paper we give two non-trivial generalizations of a classical Bernstein inequality which is apparently less known that that of Bernstein–Markov, viz.

$$|p'(x)| \leqslant k(1-x^2)^{-1/2}\,(\|p\|^2_{[-1,1]} - p^2(x))^{1/2},$$

for $x \in (-1, 1)$, where $p$ is a real polynomial of $\deg p \leqslant k$ and $\|p\|_{[-1,1]} = \sup|p|([-1, 1])$, to the case of a compact set $E$ in $\mathbb{R}^n$ with nonempty interior. Contrary to the situation where estimates for $p'(x)$ are sought on the whole compact set, we do not, in general, need any other assumptions on $E$. Our results point out connections between Bernstein's inequality and two important notions in modern polynomial approximation theory on compacta in $\mathbb{C}^n$: Siciak's extremal function and complex equilibrium measure. 

## 1

*Introduction and Statement of the Main Results.* We start with some classical inequalities for polynomials: the Bernstein–Markov inequality (see [11])

$$(1.1)\qquad |p'(x)| \leqslant k(1-x^2)^{-1/2}\,\|p\|_{[-1,1]}, \qquad \text{for} \quad x \in (-1, 1),$$

and the Bernstein inequality (see [9])

$$(1.2)\qquad |p'(x)| \leqslant k(1-x^2)^{-1/2}\,(\|p\|^2_{[-1,1]} - p^2(x))^{1/2}, \qquad \text{for} \quad x \in (-1, 1),$$

where $p$ is a real polynomial with $\deg p \leqslant k$. It is easily seen that (1.1) implies

$$(1.3)\qquad \int_{-1}^{1} |p'(x)|\, dx \leqslant \pi k \|p\|_{[-1,1]}.$$

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The main goal of this paper is to prove analogous results in the multivariate case. Let us begin with some definitions and facts from complex analysis of several variables.

If $E$ is a compact subset of $\mathbb{C}^n$ $(n \geqslant 1)$ then we define Siciak's extremal function $\Phi_E$ as follows (see [19]).

1.4. DEFINITION. $\Phi_E(z) = \sup(|p(z)|^{1/\deg p}\colon p \in \mathbb{C}[w_1, ..., w_n],\ \deg p \geqslant 1,$ $\|p\|_E \leqslant 1\}$, for $z \in \mathbb{C}^n$, where $\|p\|_E = \sup|p|(E)$. The above extremal function is also called the polynomial extremal function as opposed to the plurisubharmonic extremal function $V_E$ and its upper regularization $V_E^*$ defined as follows.

1.5. DEFINITION. $V_E(z) = \sup\{u(z)\colon u \in \mathscr{L}_n,\ u|_E \leqslant 0\}$, for $z \in \mathbb{C}^n$, where $\mathscr{L}_n$ denotes the Lelong class of plurisubharmonic functions in $\mathbb{C}^n$ (briefly, $\mathrm{PSH}(\mathbb{C}^n)$) with logarithmic growth: $\mathscr{L}_n = \{u \in \mathrm{PSH}(\mathbb{C}^n)\colon$ $\sup\{u(z) - \log(1+|z|)\colon z \in \mathbb{C}^n\} < \infty\}$.

$$V_E^*(z) = \mathop{\mathrm{lum\,sup}}_{w \to z} V_E(w), \qquad z \in \mathbb{C}^n. \tag{1.6}$$

The crucial fact is that

1.7. ZACHARIUTA–SICIAK THEOREM (see [23, 20]). $V_E = \log \Phi_E$.

For other properties of the extremal functions we refer the reader to Siciak's papers [20, 21]. We will need the notions of $L$-regularity and pluripolarity.

1.8. DEFINITION. We call a compact set $E$ $L$-regular at a point $a \in E$ if $V_E^*(a) = 0$ and we say that $E$ is $L$-regular if $E$ is $L$-regular at every point $a \in E$. It is known (see [20, 23]) that $E$ is $L$-regular if and only if $V_E$ is continuous in $\mathbb{C}^n$. Often, it is possible to use the following geometrical criterion for the $L$-regularity.

1.9. PROPOSITION (Cegrell [10], Plesniak [17], Sadullaev [18]). *Given $a \in E$, suppose that there exists an analytic mapping $h\colon [0, 1] \to E$ such that $h(0) = a$. If $V_E^*(h(t)) = 0$ for each $t \in (0, 1]$ then $V_E^*(a) = 0$.*

A pluripolar set is defined as follows.

1.10. DEFINITION. We call a set $E$ pluripolar if there exists a function $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$.

If a compact set $E$ is not pluripolar then $V_E^* \in \mathrm{PSH}(\mathbb{C}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ (see

[20]). In this case we define the complex equilibrium measure $\lambda_E$ as the value of the complex Monge–Ampère operator on the function $V_E^*$.

1.11. DEFINITION. $\lambda_E = (dd^c V_E^*)^n$. Then $\lambda_E$ is a Borel measure on $\mathbb{C}^n$ (for details we refer to Bedford and Taylor's paper [6]). We note that if $u \in \mathrm{PSH} \cap C^2(\Omega)$ then $(dd^c u)^n$ is a Borel measure defined by

$$(dd^c u)^n = n!\, 4^n \det\left[\frac{\partial^2 u}{\partial z_i\, \partial \bar{z}_j}\right] dV_n(z),$$

where $V_n$ is the Lebesgue measure in $\mathbb{C}^n$. The main properties of the complex equilibrium measure are contained in the following.

1.12. PROPOSITION [6, 22]. *If $E$ is a non-pluripolar compact set in $\mathbb{C}^n$, then*

$$\lambda_E(\mathbb{C}^n \backslash \hat{E}) = 0, \qquad \lambda_E(\hat{E}) = (2\pi)^n,$$

*where $\hat{E} = \{z \in \mathbb{C}^n \colon |p(z)| \leqslant \|p\|_E \text{ for each } p \in \mathbb{C}[w_1, ..., w_n]\}$.*

We now may formulate our main results. Let $E$ be a compact set in $\mathbb{R}^n$. We regard here $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. We need the following definition.

1.13. DEFINITION. If $E$ is a compact subset of $\mathbb{R}^n$ then we put

$$D_j^+ V_E(x) = \varliminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon e_j)$$

and

$$\mathrm{grad}_+ V_E(x) = \left(\varliminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon e_1), ..., \varliminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon e_n)\right),$$

for $x \in E$, where $\{e_1, ..., e_n\}$ is the standard orthonormal basis in $\mathbb{R}^n$.

1.1.4. THEOREM. *Let $E$ be a compact set in $\mathbb{R}^n$ with nonempty interior. Then for every $x \in \mathrm{int}(E)$ we have the following inequality for a real polynomial $p$*

$$|D_j p(x)| \leqslant (\deg p)\, D_j^+ V_E(x)(\|p\|_E^2 - p^2(x))^{1/2} \qquad \text{for} \quad j = 1, ..., n$$

*and*

$$|\mathrm{grad}\, p(x)| \leqslant (\deg p)\, |\mathrm{grad}_+ V_E(x)|\, (\|p\|_E^2 - p^2(x))^{1/2}.$$

1.15. THEOREM. *Let $E$ be a compact L-regular set in $\mathbb{R}^n$ with nonempty interior. Then the measure $\lambda_E|_{\mathrm{int}(E)}$ is absolutely continuous with respect to the Lebesgue measure and*

$$\operatorname{vol}\left(\operatorname{conv}\left\{\frac{1}{\deg p}(1-p^2(x))^{-1/2}\operatorname{grad} p(x) \colon p\in\mathbb{R}[z],\ \deg p\geqslant 1,\right.\right.$$
$$\left.\left.\|p\|_E\leqslant 1 \text{ and } |p(x)|<1 \text{ on } \mathrm{int}(E)\right)\right\}\leqslant\frac{1}{n!}\lambda(x).$$

*for almost every $x\in\mathrm{int}(E)$ (with respect to the Lebesgue measure), where $\lambda(x)\,dx=\lambda_E|_{\mathrm{int}(E)}$. If $n=1$, then the above equality reduces to*

$$\sup\left\{\frac{1}{\deg p}(1-p^2(x))^{-1/2}\,|p'(x)| \colon p\in\mathbb{R}[z],\ \deg p\geqslant 1,\right.$$
$$\left.\|p\|_E\leqslant 1 \text{ and } |p(x)|<1 \text{ on } \mathrm{int}(E)\right\}\leqslant\frac{1}{2}\lambda(x).$$

In this paper we prove only Theorem 1.14. It will be done in Section 2 while in Section 3 we discuss some special cases and examples to this theorem. The proof of Theorem 1.15, which we omit here (because it is more longer and difficult) will be published in a forthcoming paper [5] (see also [3]). However, in Section 4 we present some examples and applications of this theorem.

## 2

*Proof of Theorem* 1.14. The proof is based on the properties of the Joukowski function and its inverse. For $z\in\mathbb{C}\setminus\{0\}$ we define the holomorphic function $g(z)=(1/2)(z+1/z)$, called the Joukowski function. It is univalent on $|z|>1$ and its inverse is of the form $h(z)=z+(z^2-1)^{1/2}$, if we choose an appropriate branch of the square root. The function $\log|h|$ is subharmonic in $\mathbb{C}$ and it is well known that

$$\Phi_{[-1,1]}(z)=|h(z)|,\qquad \text{for}\quad z\in\mathbb{C}.$$

In our considerations the crucial role is played by the following equality for the above defined function $g$:

$$|g(z)+1|+|g(z)-1|=2g(|z|),\qquad z\neq 0. \tag{2.1}$$

Note that every holomorphic in $\mathbb{C}\setminus\{0\}$, non-constant solution of Eq. (2.1) has a form $g((az)^p)$ with some $a>0$ and $p \in \mathbb{N}$ (see [2]). From (2.1) it follows that

$$|h(z)| = h(\tfrac{1}{2}|z+1+\tfrac{1}{2}|z-1|), \tag{2.2}$$

for each $z \in \mathbb{C}$, where at the right-hand side we have $h(t) = t + (t^2-1)^{1/2}$ for $t \geqslant 1$ with the arithmetic root. It is easy to verify the following estimates for the function $h(t)$:

$$\sqrt{2}\,(t-1)^{1/2} - \tfrac{1}{6}(t-1)^{3/2} \leqslant \log h(t) \leqslant \sqrt{2}\,(t-1)^{1/2} \tag{2.3}$$

for every $t \geqslant 1$. An easy computation shows that the following proposition holds.

2.4. Proposition. (i) *If* $\alpha \in (-1, 1)$ *and* $\varepsilon > 0$, $\beta \in \mathbb{R}$, *then*

$$\frac{1}{\varepsilon} \log|h(\alpha + i\varepsilon\beta)| \leqslant |\beta|(1-\alpha^2)^{-1/2};$$

(ii) *If* $\alpha \in (-1, 1)$, $0 < \varepsilon \leqslant 1/2$, $\beta \in \mathbb{R}$, *and* $|\beta| \leqslant 1 - |\alpha|$, *then*

$$(1-\varepsilon)\,|\beta|\,(1-\alpha^2)^{-1/2} \leqslant \frac{1}{\varepsilon} \log|h(\alpha + i\varepsilon\beta)|.$$

Consider a real polynomial $p$ with $\|p\|_E < 1$. By well-known properties of plurisubharmonic functions (see, e.g., [12]) we have $\log|h \circ p| \in \mathrm{PSH}(\mathbb{C}^n)$ and moreover, by 2.2, we have $(1/\deg p) \log|h \circ p| \in \mathscr{L}_n$. Hence, by Definition 1.5 we obtain

$$\frac{1}{\deg p} \log|h(p(z))| \leqslant V_E(z) \tag{2.5}$$

for every $z \in \mathbb{C}^n$. Taylor's formula for $p$ now yields

$$p(x + i\varepsilon e_k) = p(x) + i\varepsilon D_k p(x) + \sum_{2 \leqslant m \leqslant \deg p} \frac{\partial^m}{\partial x^k} p(x)(i\varepsilon)^m, \tag{2.6}$$

for $1 \leqslant k \leqslant n$. It follows from Proposition 2.4 and (2.6) that

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \log|h(p(x + i\varepsilon e_k))| = |D_k p(x)|\,(1-p^2(x))^{-1/2} \tag{2.7}$$

for $x \in E$. But (2.7) together with (2.5) implies

$$|\mathrm{grad}\, p(x)| \leqslant (\deg p)\,|\mathrm{grad}_+\, V_E(x)|\,(1-p^2(x))^{1/2}. \tag{2.8}$$

If now $p$ is any real polynomial then we apply (2.8) to the polynomial $p/(\|p\|_E + \delta)$, and letting $\delta \to 0+$ completes the proof of Theorem 1.14.

2.9. *Remark.* If $E$ is a compact set in $\mathbb{R}^n$ then it follows easily that $\Phi_E(z)=\sup\{|h(p(z))|^{1/\deg p}: p\in\mathbb{R}[w_1,...,w_n], \deg p\geqslant 1, \|p\|_E\leqslant 1\}$, where $h$ denotes, as in the whole paper, the inverse function to the Joukowski function.

## 3

In this section we consider some special cases of Theorem 1.14. Let $E$ be a compact, convex, and symmetric subset of $\mathbb{R}^n$ with $\operatorname{int}(E)\neq\varnothing$. By $E^*$ we denote the dual convex set to $E$:

$$E^*=\{x\in\mathbb{R}^n: x\cdot y\leqslant 1 \text{ for each } y\in E\}.$$

It is known that

$$\Phi_E(z)=\sup\{|h(z\cdot w)|: w\in\partial E^*\},$$

for $z\in\mathbb{C}^n$ (see [14, 7]) and more precisely [1],

$$\Phi_E(z)=\sup\{|h(z\cdot w)|: w\in\operatorname{extr}(E^*)\}, \tag{3.1}$$

where $\operatorname{extr}(E^*)$ denotes the set of all extremal points of $E^*$. In the special case of $E=\bar{B}_n=\{x\in\mathbb{R}^n: x_1^2+\cdots+x_n^2\leqslant 1\}$ we have (see [15, 1])

$$\Phi_E(z)=(h(|z|^2+|z^2-1|))^{1/2}, \qquad z\in\mathbb{C}^n,$$

where $z^2=z_1^2+\cdots+z_n^2$.

An easy computation shows that

$$|\operatorname{grad}_+ V_{\bar{B}_n}(x)|=(n-1+(1-x^2)^{-1})^{1/2}\leqslant\sqrt{n}(1-x^2)^{-1/2}.$$

Thus it follows from Theorem 1.14 that for each real polynomial $p$

$$|\operatorname{grad} p(x)|\leqslant(\deg p)(n-1+(1-x^2)^{-1})^{1/2}(\|p\|_{\bar{B}_n}^2-p^2(x))^{1/2},$$

for $|x|<1$, which extends the Bernstein inequality (1.2).

Let now $f$ be any norm in $\mathbb{R}^n$ and put $E=\{x\in\mathbb{R}^n: f(x)\leqslant 1\}$. It is easy to check that $f(x)=\sup\{|x\cdot w|: w\in\operatorname{extr}(E^*)\}$. Since $E$ is compact, convex, and symmetric (with nonempty interior) it follows from (3.1) that

$$\begin{aligned}\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}V_E(x+i\varepsilon e_k)&\\ &=\sup\{|e_k\cdot w|(1-(x\cdot w)^2)^{-1/2}: w\in\operatorname{extr}(E^*)\}\\ &\leqslant f(e_k)(1-f^2(x))^{-1/2},\end{aligned} \tag{3.2}$$

if $f(x)<1$. This yields the following generalization of the Bernstein inequality:

3.3. PROPOSITION. *Let $E=\{f(x)\leqslant 1\}$, where $f$ is a norm in $\mathbb{R}^n$. Then*

$$|D_j p(x)|\leqslant (\deg p)\, f(e_j)(1-f^2(x))^{-1/2}\,(\|p\|_E^2-p^2(x))^{1/2}$$

*if $f(x)<1$, where $p$ is any real polynomial and $j=1,\ldots,n$.*

It is clear that $V_E\leqslant V_F$, if $F\subset E$. A trivial verification shows that if a compact set $E$ has nonempty interior and $x\in\operatorname{int}(E)$ then

$$|\operatorname{grad}_+ V_E(x)|\leqslant \sqrt{n}/\operatorname{dist}(x,\partial E). \tag{3.4}$$

In particular, $|\operatorname{grad}_+ V_E(x)|$ is always finite if $x$ is an interior point of $E$.

3.5. EXAMPLE. Let $S_n$ be the standard simplex in $\mathbb{R}^n$:

$$S_n=\{x\in\mathbb{R}^n\colon x_1,\ldots,x_n\geqslant 0 \text{ and } x_1+\cdots+x_n\leqslant 1\}.$$

Then (see [1]) we have $\Phi_{S_n}(z)=h(|z_1|+\cdots+|z_n|+|z_1+\cdots+z_n-1|)$ for $z\in\mathbb{C}^n$. An easy computation shows that

$$|\operatorname{grad}_+ V_{S_n}(x)|=(n(1-x_1-\cdots-x_n)^{-1}+1/x_1+\cdots+1/x_n)^{1/2}.$$

Now we will prove an interesting version of Bernstein's inequality for convex sets in $\mathbb{R}^n$. Let $E$ be a compact, convex subset of $\mathbb{R}^n$ with non empty interior. For simplicity assume that $0\in\operatorname{int}(E)$. Then the following proposition holds.

3.6. PROPOSITION (see [4]). *If $E$ is a compact, convex subset of $\mathbb{R}^n$ with $0\in\operatorname{int}(E)$ and $E^*$ is the convex dual set to $E$, then*

$$\Phi_E(z)\leqslant \inf_{d\in\operatorname{int}(E)}\ \sup_{w\in K}\left|h\left(\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}\right)\right|,\qquad \text{for}\quad z\in\mathbb{C}^n,$$

*where $K=(2/(1+|\alpha|))\operatorname{extr}(E^*)$, $\alpha=\inf\{x\cdot y\colon x\in E,\ y\in E^*\}$, and $\beta=-(1+\alpha)/(1+|\alpha|)$.*

Now, fix $x\in\operatorname{int}(E)$. Let $d'=(1/2)(x+d)$ for any $d\in\operatorname{int}(E)$. By Proposition 3.6 we obtain

$$V_E(x+i\varepsilon e_j)\leqslant \inf_{d\in\operatorname{int}(E)}\ \sup_{w\in K}\log\left|h\left(\frac{(1/2)(x-d)\cdot w+i\varepsilon e_j w}{1-|d'\cdot w+\beta|}\right)\right|.$$

Hence we get

$$D_j^+ V_E(x) \leqslant \inf_{d \in \operatorname{int}(E)} \sup_{w K} |e_j \cdot w| \, (1 - |d' \cdot w + \beta|)^{-1}$$
$$\times \left(1 - \left(\frac{(x-d) \cdot w}{2(1 - |d' \cdot w + \beta|)}\right)^2\right)^{-1/2}$$
$$= \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} |e_j \cdot w| \left\{(1 - |d' \cdot w + \beta|)^2 - \left(\frac{1}{2}(x-d) \cdot w\right)^2\right\}^{1/2}$$
$$\leqslant \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} |e_j \cdot w| (1 - |x \cdot w + \beta|)^{-1/2} (1 - |d \cdot w + \beta|)^{-1/2}$$
$$\leqslant \sup_{w \in K} (|e_j \cdot w| / |w|)(\operatorname{dist}(x, \partial E))^{-1/2}$$
$$\inf_{d \in \operatorname{int}(E)} (\operatorname{dist}(d, \partial E))^{-1/2}.$$

(Here $e_1, ..., e_n$ denotes the standard orthonormal basis in $\mathbb{R}^n$.) The above inequality yields the following

3.7. THEOREM. *Let $E$ be a convex, compact subset of $\mathbb{R}^n$ and such that $0 \in \operatorname{int}(E)$. Then for every real polynomial $p$ we have the Bernstein inequality*

$$|D_j p(x)| \leqslant (\deg p) \, M(\operatorname{dist}(x, \partial E))^{-1/2} (\|p\|_E^2 - p^2(x))^{1/2},$$

*for $x \in \operatorname{int}(E)$, $j = 1, ..., n$, where the constant $M$ is equal to*

$$M = \max_{j=1,...,n} \sup_{w \in K} (|e_j \cdot w| / |w|) \inf_{d \in \operatorname{int}(E)} (\operatorname{dist}(d, \partial E))^{-1/2}.$$

3.8. *Remark.* If $E$ is any compact, convex subset of $\mathbb{R}^n$ with nonempty interior and $b \in \operatorname{int}(E)$, then $0 \in \operatorname{int}(E - b)$ and we may apply Theorem 3.7 to the subset $E - b$. This gives the Bernstein inequality for the set $E$ with a different constant $M$ than that of Theorem 3.7.

3.9. *Remark.* We shall say that a compact subset $E$ of $\mathbb{R}^n$ (with nonempty interior) has Bernstein's property if for every real polynomial $p$ the following inequalities hold:

$$|D_j p(x)| \leqslant (\deg p) \, M(\operatorname{dist}(x, \partial E))^{-\mu} (\|p\|_E^2 - p^2(x))^{1/2}, \qquad \text{for} \quad x \in \operatorname{int}(E),$$

$j = 1, ..., n$, where $M > 0$ and $0 < \mu < 1$. Observe that every compact subset of $\mathbb{R}^n$ with nonempty interior satisfies the above inequality with the constant $\mu = 1$. We conjecture that every fat ($E \subset \overline{\operatorname{int}(E)}$) compact subset of $\mathbb{R}^n$

that preserves Bernstein's inequality with $\mu < 1$ has the following Markov property: There exists a constant $M$ such that for every real polynomial $p$,

$$\|D_j p\|_E \leqslant M(\deg p)^\alpha \|p\|_E, \qquad j = 1, ..., n,$$

with $\alpha = 1/(1-\mu)$.

We also note that the above conjecture is true in the case of compact, convex sets (see [16]).

## 4

In this section we will prove the following two estimates for real polynomials resulting from Theorem 1.15.

4.1. THEOREM. *Let $E$ be an L-regular compact subset of $\mathbb{R}^n$ with noempty interior. Then for almost every $x \in \operatorname{int}(E)$ the following inequality holds*

$$|\operatorname{grad} p(x)| \leqslant 2^{-n}(\deg p)\, d(x)\, \lambda(x)(\|p\|_E^2 - p^2(x))^{1/2},$$

*for a real polynomial $p$, where $\lambda(x)$ is the density on* $\operatorname{int}(E)$ (*with respect to the Lebesgue measure*) *of the complex equilibrium measure and*

$$d(x) = [(d_1^2 - x_1^2) \cdot \cdots \cdot (d_n^2 - x_n^2)((d_1^2 - x_1^2)^{-1} + \cdots + (d_n^2 - x_n^2)^{-1})]^{1/2},$$

*with $d_j = \sup|z_j|(E)$, $j = 1, ..., n$.*

4.2. THEOREM. *If $E$ is a fat ($E \subset \overline{\operatorname{int}(E)}$) compact subset of $\mathbb{R}^n$ with zero Lebesgue measure on $\partial E$, then*

$$\int_E |\operatorname{grad} p(x)|\, dx \leqslant \pi^n(\deg p)\, d(0)\, \|p\|_E,$$

*for any real polynomial $p$, where $d(x)$ is defined in Theorem* 4.1.

4.3. *Proof of Theorem* 4.1. Without loss of generality we can assume $|p(x)| < \|p\|_E$ for $x \in \operatorname{int}(E)$. From Theorem 1.15 it follows that

$$\begin{aligned} n!\, &\operatorname{vol}(\operatorname{conv}\{\pm(\|p\|_E^2 - p^2(x))^{-1/2} \operatorname{grad} p(x), \\ &\quad \pm(d_1^2 - x_1^2)^{-1/2} e_1, ..., \pm(d_i^2 - x_i^2)^{-1/2} \hat{e}_i, ..., \pm(d_n^2 - x_n^2)^{-1/2} e_n\}) \\ &= n!\, 2^n |D_i p(x)|\, (\|p\|_E^2 - p^2(x))^{-1/2} (d_1^2 - x_1^2)^{-1/2} \cdot \cdots \cdot (d_n^2 - x_n^2)^{-1/2} \\ &\quad \cdot (d_i^2 - x_i^2)^{1/2} \cdot \operatorname{vol}(\operatorname{conv}\{0, e_1, ..., e_n\}) \\ &= 2^n |D_i p(x)|\, (\|p\|_E^2 - p^2(x))^{-1/2} (d_1^2 - x_1^2)^{-1/2} \cdot \cdots \cdot (d_n^2 - x_n^2)^{-1/2} \\ &\quad \cdot (d_i^2 - x_i^2)^{1/2} \leqslant \lambda(x) \qquad \text{for } j = 1, ..., n \text{ and for almost every } \\ &\qquad\qquad x \in \operatorname{int}(E). \end{aligned}$$

Here $\hat{e}_i$ denotes that the $\pm(d_i^2 - x_i^2)^{1/2} e_i$ is missing. Combining these $n$ inequalities we obtain 4.1.

4.4. *Proof of Theorem* 4.2. Given a fat compact subset of $\mathbb{R}^n$ define

$$F_k = \left\{ x \in E \colon \operatorname{dist}(x, \partial E) \geqslant \frac{1}{k} \right\}$$

and

$$E_k = \bigcup_{x \in F_k} \bar{B}(x, 1/(k+1)), \text{ for } k \in \mathbb{N},$$

where $\bar{B}(x, r)$ denotes the closed euclidean ball with center at $x$ and radius $r$. We have $E_k \subset E_{k+1}$ and $\operatorname{int}(E) = \bigcup \operatorname{int}(E_k)$. Moreover, the sets $E_k$ are compact, fact, and (by 1.9) $L$-regular. By 4.1 and 1.12 we obtain

$$\int_{\operatorname{int}(E_k)} |\operatorname{grad} p(x)| \, dx \leqslant \pi^n (\deg p) \, d(0) \, \|p\|_E$$

and letting $k \to \infty$ gives 4.2.

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*Note added in proof.* In Theorem 4.2 it suffices to assume the set $E$ is compact in $\mathbb{R}^n$. This follows by the fact that there exists a sequence $E_n \supset E_{n+1}$ of compact fat subsets of $\mathbb{R}^n$ such that $E = \bigcap E_n$ and each $E_n$ has zero Lebesgue measure on $\partial E$.

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